MEM6810 Engineering Systems Modeling and Simulation
工程系统建模与仿真

Theory Analysis

Lecture 9: Output Analysis II: Comparison

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Spring 2023 (full-time)

上海交通大学 SHANGHAI JIAO TONG UNIVERSITY | 董浩云智能制造与服务管理研究院 CY TUNG Institute of Intelligent Manufacturing and Service Management (中美物流研究院) (Sino-US Global Logistics Institute)

Contents

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- It is one of the most important uses of simulation.

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  - the actual differences on the expected performance of system designs?
  - or the random errors in the simulation outputs?
- The comparison can be classified into two types:
  - Two system designs: using confidence interval of the difference.
  - Multiple (more than two) system designs: selection of the best.

Comparison of Two Designs

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System	*Replication* 1	2	$\cdots$	R_i	*Sample Mean*	*Sample Variance*
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- Approximate $1-\alpha$ CI: $\bar{Y}_1 - \bar{Y}_2 \pm t_{v,\,1-\alpha/2} \times \mathrm{s.e.}(\bar{Y}_1 - \bar{Y}_2)$.
    - $\mathrm{s.e.}(\bar{Y}_1 - \bar{Y}_2)$ is the estimator of standard error of $\bar{Y}_1 - \bar{Y}_2$; see more details about this quantity and v later.

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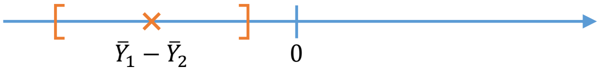

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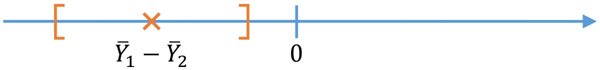

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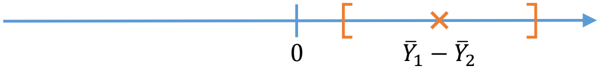

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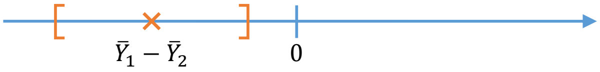

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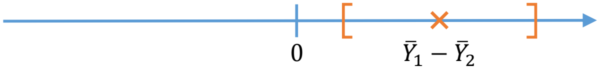
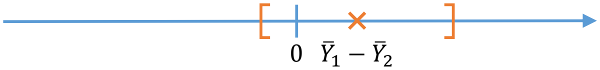


- Case 3 – No strong evidence that one is larger than the other:



  - It does not imply $\theta_1 = \theta_2$!

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- The first two cases are conclusive.
- If in case 3, then we increase the number of replications R_1 and/or R_2, after which the CI would likely shift, and definitely shrink in length.
- We will shrink the CI until case 1 or 2 is achieved, or the confidence interval is so narrow, which suggests that we do not need to separate them.

- For the comparison of performance of two designs, there is an important distinction between
  - *statistically significant difference* (统计意义上的显著区别);
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  - Is the true difference $|\theta_1 - \theta_2|$ large enough so it is worthwhile to separate them?

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- Confidence intervals do not answer the question of practical significance directly.
  - Instead, they bound, with probability $1 - \alpha$, the true difference $\theta_1 - \theta_2$ within the range $\bar{Y}_1 - \bar{Y}_2 \pm t_{v,\,1-\alpha/2} \times \mathrm{s.e.}(\bar{Y}_1 - \bar{Y}_2)$.
  - Whether a difference within these bounds is practically significant depends on the particular problem.

- Independent sampling means that different random number streams are used to simulate the two systems.
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$$\mathrm{s.e.}(\bar{Y}_1 - \bar{Y}_2) = \sqrt{\frac{S_1^2}{R_1} + \frac{S_2^2}{R_2}}. \quad (1)$$

- The $1-\alpha$ CI is approximated by

$$\bar{Y}_1 - \bar{Y}_2 \pm t_{v,\,1-\alpha/2} \times \mathrm{s.e.}(\bar{Y}_1 - \bar{Y}_2). \qquad (2)$$

where $\mathrm{s.e.}(\bar{Y}_1 - \bar{Y}_2)$ is given in (1), and the degree of freedom v is

$$v = \frac{[S_1^2/R_1 + S_2^2/R_2]^2}{[S_1^2/R_1]^2/(R_1-1) + [S_2^2/R_2]^2/(R_2-1)}.$$

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- The approximated CI (2) is called the *Welch confidence interval* (Welch 1938).
    - Sometimes, people will round v to integer for convenience.

- If $R_1 = R_2 = R$, or we are willing to discard some observations from the system design on which we actually have more data, we can pair Y_{r1} with Y_{r2} to define $Z_r = Y_{r1} - Y_{r2}$, for $r = 1, \ldots, R$.

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- Approximate $1 - \alpha$ CI:

$$\bar{Z} \pm t_{R-1,\,1-\alpha/2}\frac{S}{\sqrt{R}}. \tag{5}$$

- Common Random Numbers (CRN, also known as correlated sampling): For each replication, the same random numbers are used to simulate both systems.
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- CI is still computed via (4) and (5), but the width will be smaller $\Longrightarrow$ higher precision.

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  - The random numbers must be synchronized: each random number used in one model for some purpose should be used for the same purpose in the other model.
  - E.g., if the ith random number is used to generate a service time at work station 2 for the 5th arrival in model 1, the ith random number should be used for the very same purpose in model 2.

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- The CRN idea is also used when we validate simulation model via input-output transformation, where we prefer to compare the model and actual system under the same historical input, rather than generate the input from input model.

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- The first three can be achieved by **simultaneous** construction of confidence intervals, whereas the last by some **selection approaches**.
- From now on, without loss of generality, let's *assume the best* θ_i *is the largest one*.

- Assumption 1: For each design i with mean performance θ_i, the noisy output $Y_{ri} \sim \mathcal{N}(\theta_i, \sigma_i^2)$, for $r = 1, 2, \ldots$.
- Assumption 2: No CRN is used, i.e., Y_{ri} is independent of Y_{rj} for $i \neq j$.
- Assumption 3 (**indifference-zone**): The gap between the largest θ_i and the second largest θ_i is at least δ, *a value known to us*.
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- Bechhofer (1954) first developed a selection procedure, which can ensure the probability of correct selection (PCS):

$$\mathbb{P}\{\text{select the largest } \theta_i\} \geq 1 - \alpha, \tag{7}$$

under Assumptions 1-4, where α is a user specified value and $1 - \alpha > 1/k$.

- Bechhofer's Procedure
  1. Calculate a constant h, which satisfies

$$\mathbb{P}\{Z_i \leq h, \ i = 1, 2, \ldots, k-1\} = 1 - \alpha, \tag{8}$$

     where $(Z_1, Z_2, \ldots, Z_{k-1})^\intercal$ has a multivariate normal distribution with means 0, variances 1, and common pairwise correlations $1/2$.
  2. For $i = 1, \ldots, k$, let

$$n_i = \left\lceil \frac{2h^2\sigma_i^2}{\delta^2} \right\rceil. \tag{9}$$

  3. For $i = 1, \ldots, k$, run n_i replications for design i and calculate

$$\bar{Y}_i = \frac{1}{n_i} \sum_{r=1}^{n_i} Y_{ri}.$$

  4. Select the design with the largest sample mean $\bar{Y}_i$ as the best.

Proof.

Without loss of generality, assume $\theta_k \geq \theta_{k-1} \geq \cdots \geq \theta_1$. Then Assumption 3 says, $\theta_k - \theta_{k-1} \geq \delta$, which implies that

$$\theta_k - \theta_i \geq \delta, \ i = 1, \ldots, k-1. \tag{10}$$

Proof.

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Now we only need to check that $\boldsymbol{Z} = (Z_1, Z_2, \ldots, Z_{k-1})^\mathsf{T}$ indeed has a multivariate normal distribution with means 0, variances 1, and common pairwise correlations $1/2$ (except for some rounding error).

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Hence, by (8) and (11), $\mathbb{P}\{\text{select } k\} \geq 1 - \alpha$. ■

- Assumption 3 (indifference-zone) can be **relaxed** by *softening* the selection target to probability of good selection (PGS):

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- Procedures like Bechhofer's or Rinott's are simple to implement, but the efficiency may be low.
  - The designed sample size (or, replication number), n_i, may be larger than necessary (too conservative).

- More sample efficient procedures should be in a sequential manner.
  - Take observations sequentially, i.e., one at a time.
  - Eliminate designs from continued sampling when it is statistically clear that they are inferior.
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- Paulson (1964) proposed fully sequential procedures, which can guarantee the PCS in (7), under Assumptions 1-3 and (a) *common known* variance or (b) *common unknown* variance.

- Suppose $\sigma_1^2 = \sigma_2^2 = \cdots = \sigma_k^2 = \sigma^2$ and σ^2 is known (*common known* variance).
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- Paulson's Procedure

1. Let $0 < \lambda < \delta$ (a good choice is $\lambda = \delta/2$), and
$$a = \ln\left(\frac{k-1}{\alpha}\right)\frac{\sigma^2}{\delta - \lambda}.$$
Let $I = \{1, 2, \ldots, k\}$ and $r = 0$.
2. Let $r \leftarrow r + 1$. Take one observation from each alternative in I and compute $\bar{Y}_i(r)$, $\forall i \in I$.
3. Let $I^{\text{old}} = I$ and
$$I = \left\{\ell \in I^{\text{old}} : \bar{Y}_\ell(r) \geq \max_{i \in I^{\text{old}}} \bar{Y}_i(r) - \max\{0, a/r - \lambda\}\right\}.$$
If $|I| > 1$, then go to Step 2; otherwise, select the alternative left in I as the best.

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- Commercial simulation software, Simio, implements the $\mathcal{KN}$ procedure of Kim and Nelson (2001) as an Add-In, to help user to select the best scenario.

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- Ranking and Selection (R&S) problem was first introduced in the 1950s by the statistics community:
  - rank all alternatives
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    - rank all alternatives
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- Existing procedures for R&S (selection of the best) problems:
    - frequentist
    - Bayesian

- Frequentist procedures typically aim to deliver the PCS or PGS; see Kim and Nelson (2006) for a review:
  - two-stage procedures: Bechhofer (1954), Rinott (1978)
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- Bayesian procedures often allocate samples to each alternative either to maximize the Bayesian posterior PCS or to minimize the expected opportunity cost; see Chen et al. (2015) for a review:
  - optimal computing budget allocation: Chen et al. (2000), He et al. (2007)
  - value of information: Chick and Inoue (2001), Chick et al. (2010)
  - knowledge gradient: Frazier et al. (2008), Frazier et al. (2009)
  - economics of selection procedures: Chick and Gans (2009), Chick and Frazier (2012)

- Emerging research problems that expend classical R&S from different perspectives; see Hong et al. (2021) for a review:
  - large-scale R&S using parallel computing
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  - **Simulation Optimization (or called Optimization via Simulation)**

- R&S Problem vs Multi-Arm Bandit (MAB) Problem: